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An Outline of Possible In-course Diagnostics for Functions

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ABSTRACT The present paper focuses on in-course sample diagnostic questions relating to functions for students who choose to take a course on differential calculus, in a South African context. However, the diagnostic questions should be useful at many places worldwide. The researchers formulated learning outcomes and in-course diagnostic questions for the technical knowledge and skills required for the section on functions, in the context of differential calculus. The questions were designed to check the relevant learning outcomes that they detected for the section on functions as informed by the literature review and conceptual framework. The learning outcomes and formulation of diagnostic outcomes, although based on a number of assumptions, should improve the performance of students.

INTRODUCTION

At the University of KwaZulu-Natal (UKZN) relatively low pass rates for the first year differential calculus (Math130) module was of concern. A previous paper (Maharaj and Wagh 2014) looked at the pre-course diagnostics for differential calculus. In the present paper the focus is to outline an in-course diagnostics for functions, as required for differential calculus. This aims at assessing the strengths and shortcomings of technical knowledge and skills, of the student. The diagnostic tests would not be for grading students but rather to provide feedback on their strengths and weaknesses with regard to content and skills on functions. The researchers formulated detailed learning outcomes for the topic on functions for the first year differential calculus course offered at the UKZN.

The research questions focused on were: *What are the expected learning outcomes with regard to the section on functions for differential calculus? How could in-course diagnostics on functions for differential calculus be developed?*

Objectives

There are two main objectives: (1)Formulating learning outcomes for the topic on functions, as relevant to the study of differential calculus. (2) Use those learning outcomes to design sample in-course diagnostic questions for the section on functions.

Review of Literature

In the researchers' opinion in-course diagnostic testing is an integral part of assessment. Diagnostic tests should be informed by the suitable formulation of learning outcomes for a module or course. In any first course on calculus, the building block for the concept of a derivative is the function concept. The literature review focuses on (1) learning outcomes and their area of application, and (2) functions.

Learning Outcomes and Area of Application

Adams (2006) in his discussion of learning outcomes notes that : (a) these are statements of what a learner is expected to know, understand and/or be able to demonstrate at the end of a period of learning; (b) they indicate the results of learning; (c) they are usually defined in terms of a mixture of knowledge, skills, abilities, attitudes and understanding that an individual will attain as a result of his or her successful engagement in a particular set of higher education experiences; (d) in reality, they represent much more than this since they could exemplify a particular methodological approach for the expression and description of the curriculum (modules, units and qualifications), and also levels, cycles and subject benchmark statements. The researchers' study of the work of Adam (2006) indicates that learning outcomes: (a) have multiple roles; (b) have advantages for educational reform with regard to pedagogy, assessment and quality assurance; and (c) could shift the delivery of the curriculum to one that is more learner centred.To an extent, the latter two issues could be addressed by the formulation of in-course diagnostic tests, informed by relevant learning outcomes. The areas of application of learning outcomes, as relevant to this paper, are presented at the end of this sub-section. It is worth noting that learning outcomes should inform the formulation of diagnostic test questions. By diagnostic questions the researchers mean questions that check the strengths and weaknesses of students related to the correct assimilation of concepts or proper command of skills at a required level.

Betts et al. (2011) reported that a number of institutions in the State of California use pre-course and to some extent in-course diagnostic testing, and their preliminary results show improvement in student performance. In those institutions the California Mathematics Diagnostic Testing Project (MDTP) tests were used. The Learning and Technology Support Network Maths Team Project (2003) reported that many institutions in the United Kingdom use diagnostic testing. Some use paper based tests (for example Cardiff University, Coventry University, Manchester Metropolitan University) and others (for example University of Bristol, Brunel University, Keele University) use computer based tests. Those were pre-course diagnostic tests.

The following is adapted from Adam (2006). The researchers give the area of application of outcomes followed by their features and attributes.

Unit, Module or Course: These are learning outcomes used at the level of the unit or module as statements that identify what a successful learner will be able to know, understand or be able to do. The features and attributes are: (1) Concerned with the achievements of the learner. (2) Differ from 'aims' that indicate the intentions of the teacher. (3) Directly link to a teaching strategy for the effective delivery of the learning outcomes. (4) Directly link to an assessment strategy and appropriate assessment criteria. (5) Are developed in a context of a wide range of internal and external reference points and influences.

Assessment, Diagnostic Tests: At the level of the unit or module, learning outcomes can be used to express the criteria that establishes the standards of achievement. The features and attributes are: (1) Description of what the learner is expected to do to demonstrate that the learning outcome has been achieved. (2) Description informs the framing of appropriate questions on the content for the unit or module. (3) Questions normally written at a threshold level and distinguish the pass and fail threshold.

Functions

The researchers note that functions are a central part of the pre-calculus and calculus curriculum (Tall 1997). Maharaj (2013) noted that this was supported by a study of first year calculus prescribed textbooks (for example the textbooks by Stewart 2009; Lial et al. 2008) at the University of KwaZulu-Natal, which revealed that a good grounding of the following is a prerequisite for introducing the derivative concept: [a] *algebraic manipulations*, and [b] *functions, including their symbolic and graphical representations*. The implication is that before the concept of the derivative of a function is introduced students should have adequately established algebraic manipulation skills and understanding of the concept of a function (Maharaj 2013). Maharaj and Wagh (2014) focused on *algebraic manipulations* in the context of pre-course diagnostics for differential calculus. This paper focuses on *functions, including their symbolic and graphical representations* in the context of in-course diagnostics for differential calculus. The focus is also on the calculus related concepts of functions, particularly those visible from graphical representations. The reason for this approach is based on the literature review of the paper that the researchers now summarise.

Previous studies on functions in the context of calculus demonstrate that: (a) Generally, students exhibit a predominant reliance on the use of and the need for algebraic formulae when dealing with the function concept (Breidenbach et al. 1992; Asiala et al. 1997; Akkoc and Tall 2005; Maharaj 2013). This seems to have resulted in an over-reliance by students on algebraic formula as a means of understanding concepts in calculus (Berry and Nyman 2003; Habre and Aboud 2006; Shepherd et al. 2012). (b) For students to obtain a holistic understanding the ability to identify and represent the same mathematical idea or concept in different representations

(Habre and Abboud 2006; Haciomeroglu et al. 2010); like numerical, graphical and algebraic; is suggested. (c) In particular the graphical sense of calculus needs further emphasis if we want students to develop *a feel of calculus* (Berry and Nyman 2003). If these are accepted then the ability to make connections relating to the calculus properties of graphs (for example intervals of increase or decrease) should build a better understanding of the underlying graphical concepts of calculus, not just knowledge of rules. (d) Learning about functions and their graphical interpretation with suitable connections could ensure that students effectively learn concepts in calculus (Mahir 2010). Such an approach should be effective in allowing students to construct correct concept images which in turn could assist their successful interpretation of information from graphs. Mahir (2010) argued that adding this visual dynamic to the analytical dimension of the teaching of the concept should enable students to develop better understanding of concepts and the interrelationship between the visual and the analytical. (e) If one views algebraic thinking as analytical and graphical thinking as visual then Haciomeroglu et al. (2010) proposed a balanced reliance on verbal-logical (analytical) and visual-pictorial (visual) processes. They caution that an over-reliance on either of these poses a danger of one-sidedness in students' mathematical development. This could consequently compromise students' ability to successfully find solutions to problems in mathematics. (f) Other studies emphasized the importance of the concept of function composition in the understanding of the chain rule (Clark et al. 1997; Hassani 1998; Cottrill 1999; Maharaj 2013; Jojo et al. 2013), which is a technique used to find the derivative of functions, in whose structures other functions are embedded; for example $f(x) = \sqrt{x^3+1}$. Here, we can conceptualize $f(x)$ as a composition of two functions, for example $f(x) = g(h(x))$ where $g(x)=\sqrt{x}$ and $h(x) = x^3+1$.

The above imply that differential calculus concepts (for example intervals of increase or decrease and concavity) should be related to observations that focus on the key features of functions in the context of their symbolic and graphical representations. These are the main aspects that informed the conceptual framework and methodology for the study.

Conceptual Framework

The conceptual framework for outlining possible in-course diagnostics on functions for differential calculus was guided by the literature review and the following principles:

1. There is a conceptual hierarchy in the body of mathematics.
2. To study mathematics students should have good work habits and they should know what is meant by these. [Focused on in the paper by Maharaj and Wagh (2014).]
3. It is important for the learning outcomes of a unit or module, to be clearly documented.
4. For effective learning to occur it is not good enough for only the instructor (teacher, lecturer or tutor) to be aware of the technical knowledge outcomes of a course/module.
5. Students should know explicitly at the outset the learning outcomes expected of them.
6. Instructors should use the documented learning outcomes to formulate suitable diagnostic questions, for students.
7. When students attempt the diagnostic questions there should be provisions for remedial activity, to overcome their identified shortcomings.

METHODOLOGY

The literature review, conceptual framework and study of the aims and content for the *Introduction to Calculus* module offered at UKZN informed this. The researchers first looked at the aim and content as indicated in the handbook of the Faculty of Science and Agriculture (2010) that was in the public domain. These are indicated below; the parts in italics are the researchers' emphasis:

Aim

Aim was to introduce and develop the Differential Calculus as well as the fundamentals of proof technique and rudimentary logic.

Content

Content included Fundamental Concepts : elementary logic, proof techniques. Differential Calculus - *Functions, graphs and inverse functions*, limits and continuity, the derivative, techniques of differentiation, applications of derivatives, anti-derivatives.

Further, the researchers used their experience relating to teaching at secondary and tertiary education institutions to document learning outcomes and diagnostic questions on functions.

1. Expected student work habits and pre-course diagnostics. Students should be aware of the work habits that their instructors (lecturers, tutors) expect from them. The researchers documented such outcomes and diagnostic questions that in their opinion give the characteristics of the expected work habits of successful students. [A focus of another paper by Maharaj and Wagh (2014).]
2. Pre-course (pre-requisites) outcomes and diagnostic questions. Based on the conceptual hierarchy of mathematics and common errors of past students the researchers formulated such outcomes and diagnostic questions. [Focus of the paper by Maharaj and Wagh (2014).]
3. In-course learning outcomes for functions. The researchers formulated such outcomes for the *Introduction to Calculus* module by studying its aim and content (as indicated above), and also past assessment questions. [The focus of this paper.]
4. In-course diagnostics for functions. The researchers used the learning outcomes identified to formulate questions on course content for functions. The planning of the diagnostic questions were influenced by the literature review, which suggested that there should a deliberate integration of differential calculus related concepts to observations on the key features of functions, in the context of their symbolic and graphical representations.

FINDINGS AND DISCUSSION

The researchers present the learning outcomes for functions followed by their formulations of the sample diagnostic questions that check for student attainment of those learning outcomes.

Learning Outcomes

The researchers' study led to the formulation of 22 learning outcomes for functions in the context of the Introduction to Calculus module. Where necessary they unpacked some of them by making use of bullets at the relevant places.

The researchers expect students to be able to:

1. identify subsets
2. write and recognize Cartesian product of two sets
3. write and recognize the arrow or graphical representation of a relation of an element of a set
4. read and interpret a graph
    - ➢ recognize functions or non-functions that are represented graphically
    - ➢ recognize the domain and range of functions represented graphically
    - ➢ identify the associated defining equation for a given standard graphical representation
    - ➢ identify the graph associated with a defining equation (include discontinuous functions; likestep functions)
    - ➢ recognize a standard set of observations of a function
        - o Identify if a function is even, odd or neither
        - o Domain and range
        - o Number of times the graph intersects the x-axis
        - o The number of times the function changes sign
        - o Continuity of the graph
        - o Intervals of constancy or increase or decrease
        - o Rate of change of increase or decrease from the function's graph
        - o Number of points where the graph turns from increasing to decreasing and vice versa
        - o Number of points of inflection from the function's graph
        - o Number of asymptotes
        - o Local extrema and absolute extrema
        - o Saddle points if the function is of more than one variable
- ➢ use the defining equation of a standard function to give key features of the associated graphical representation (including discontinuous graphs)
    - o Constant functions
    - o Step functions
    - o Polynomial functions
    - o Rational functions
    - o Trigonometric functions
    - o Exponential functions

 - o Logarithmic functions
 - o Inverse trigonometric functions

5. recognize the effect of vertical translations of a standard graph on its associated defining equation
6. recognize the effect of horizontal translations of a standard graph on its associated defining equation
7. manipulate the defining equation of a standard function with regard to vertical translations of the associated graph
8. manipulate the defining equation of a standard function with regard to horizontal translations of the associated graph
9. graphically represent the effect of a reflection of a graph about an axis
10. identify the change in the associated defining equation of a function when the graph of the function is reflected about an axis
11. truncate the graph of a function by restricting the domain of the defining equation
12. recognize multiple ways of combining equations of two functions to create a new defining equation.
    - adding functions
    - subtracting functions
    - multiplying functions
    - dividing functions
    - raising a function to a power of another function
    - composing functions
    - paste-ing together the graphs of functions. [paste-ing: translating the graph of one function, truncating the graphs of both functions at an intersection point after the translation, restricting the domain of parts of the graphs]
13. use the defining equation of two functions to form new functions
14. decompose a composite function into its constituent simple functions
15. recognize and state the key features of polynomial functions
16. recognize rational functions
17. determine the key features of a rational function
18. determine the key features of a function of the type $y=\sqrt{f(x)}$ where $f(x)$ is a polynomial function
19. recognize the inverse of a function
20. determine the defining equation of the inverse of a function
21. determine the key features of an inverse of a function for a given function
22. graphically represent the inverse of standard functions

Sample Diagnostic Questions

The sample diagnostic questions formulated by the researchers were based on the above learning outcomes and are indicated as Diagnostics1 to 7. These in-course diagnostic questions for functions concerns the following areas: (a) subsets and Cartesian product of two sets, (b) reading and interpreting representations of relations and functions, (c) interplay of representations and associated defining equations of functions, (d) effect of translations, reflections and truncation on the defining equations and representations of functions, (e) various ways of combining defining equations of two functions to create new functions, and (f) key features of various standard functions. In the questions formulated the researchers tried to incorporate as many of these (that were informed by the learning outcomes identified and the literature review), as often as they could.

Subsets, Cartesian Products, Relations, Functions and Non-functions

The diagnostic questions formulated, from the identified learning outcomes, are indicated in Diagnostics 1. Note that some of the diagnostic questions require the student to be able to graphically represent a relation and also be able to deduce whether a given graph represents a function.

Diagnostics 1: Diagnostic questions on subsets, Cartesian products, relations, functions and non-functions

1. List all subsets of {1; 2}.
Answer: ϕ or { }, {1}, {1;2}

2. Write the Cartesian product of the set {1; 2} with itself.
Answer: {(1, 1); (1, 2); (2, 1); (2, 2)}

3. For real numbers a and b is related to b if $b^2=a$ Give a graphical representation of this relation in the context of the number a = 4.
Answer: See Figure 1.

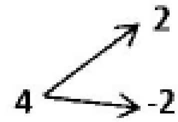

Fig. 1.

4. Does the graph in Figure 2 represent a function or a non-function?

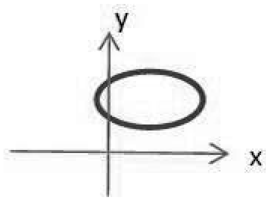

Fig. 2.

Answer: Non-function

5. Does the graph in Figure 3 represent a function or a non-function?

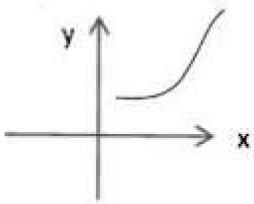

Fig. 3.

Answer: Function

6. Does the graph in Figure 4 represent a function or a non-function?

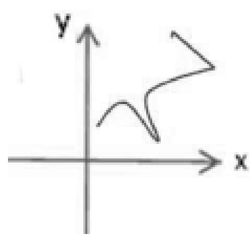

Fig. 4.

Answer: Non-function

Polynomial Functions

Diagnostics 2 and 3 indicate the diagnostic questions that were formulated, based on the identified learning outcomes, in the context of polynomial functions which are covered at school level. There are questions that require students to identify from the graphical representation of such functions their defining equations (see Diagnostics 2). Further,question 3 (in Diagnostics 2) includes the observation of relevant calculus concepts; point of inflection, decreasing over, concave up, concave down.

Diagnostics 2: Diagnostic questions and answers for polynomial functions

1. Study the graph in Figure 5 of a linear function, for which the domain is restricted. Then identify the restricted domain, range and the defining equation for this graph.

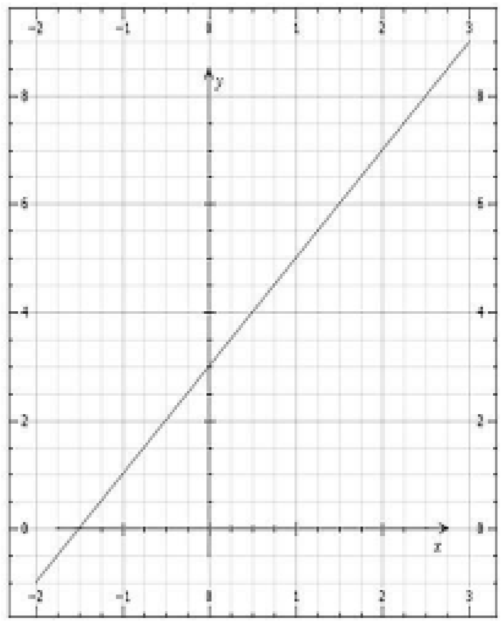

Fig. 5.

Answers: Domain is $2 \leq x \leq 3$, *[-2,3]*. Range is-1$\leq$ y $\leq$ $\leq$ 9, [-1,9]. Defining equation is y = 2x+3.

2. Study the graph in Figure 6 for which the domain is restricted. Identify the restricted domain, range, intercepts, turning point and the defining equation for this graph.

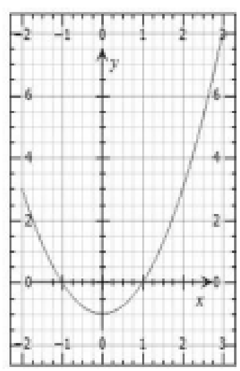

Fig. 6.

Answers:Domain is-2 $\leq x \leq 3$ [-2,3]. Range is-1 < y < 8, [1,8] y-intercept is -1; x-intercepts are -1 or 1. Turning point is (0, -1). Defining equation is $y = x^2$-a.

3. Study the graph in Figure 7 which is the result of a reflection and translation of the graph of a standard function. Identify the defining equation of the standard function and its point of inflection. For the illustrated graph find the defining equation, the interval(s) over which it is increasing/decreasing, the point of inflection and the intervals of concavity.

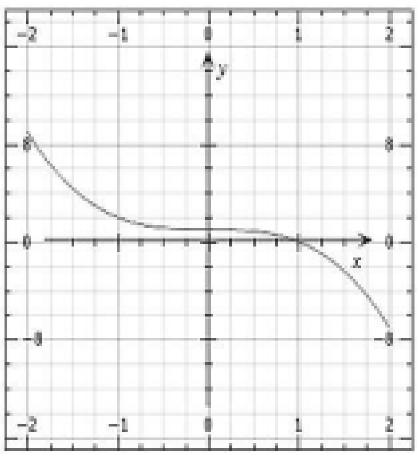

Fig. 7.

Answers: Standard function: $y=x^3$, point of inflection (0,0); Illustrated graph: $y= -x^3$, decreasing over *(-x, x)*, point of inflection is (0,1), concave down on (0,$^{(x)}$, concave up on *(-x, 0)*.

Some of the questions require students to relate the effect of reflections and/or translations of a given graph to its defining equation (see Diagnostics 2, question 3; Diagnostics 3, question 2) and related calculus concepts. The intention was for students to be able to make such observations on the effect of a translation or reflection to calculus related features including local extrema, from the graphical representation of a given function (for example, question 1 in Diagnostics 3). The researchers' thinking was that exposing students to such diagnostic questions would enable them to form an awareness of what is meant by key features of a function. The assumption is that the exposure to such questions would improve the observational capacity of students. This assumption has to be investigated, possibly in another paper.

Also note that in some of the questions (for example see Diagnostics 3, question 1) the information deduced from the graph may not be accurate. For example x-intercepts, regions of decrease, point of inflection and local extrema. In such cases it is expected that students use their relevant knowledge and skills to perform the required calculations. It requires the student to use calculus related calculations to verify information that is not clear in the graphical representation of a function when $y = x^2\text{-}2^{x2}+1$ given its defining equation, in this case .The researchers' thinking was that a deliberate emphasis on the graphical sense of calculus could lead our students to develop *a feel of calculus* (Berry and Nyman 2003).

Diagnostics 3: Diagnostic questions, effect of a reflection about an axis on a given function

1. Study the graph of $y = x^{3\text{-}}2^{x2} +1$ in Figure 8 and identify its key features.

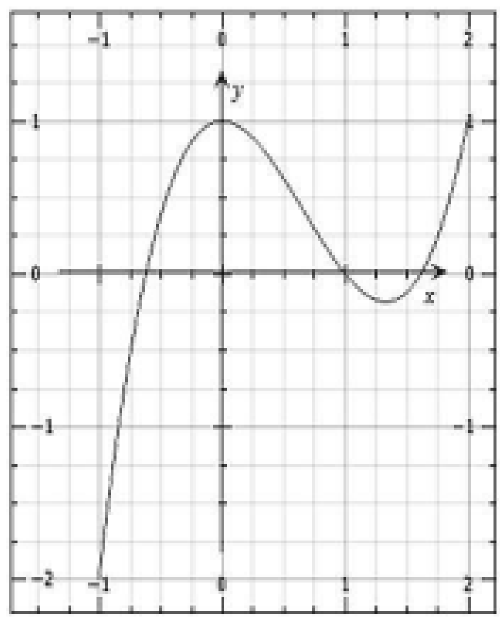

Fig. 8.

Answers: Domain is *(-x, x)* and range is *(-x, x)*; y-intercept is 1; x- intercepts are:

$x = 10$ or $x = \frac{\sqrt{1\text{-}5}}{2}$ or $\frac{\sqrt{5+1}}{2}$. Turning points (0,1) and $\left(\frac{4}{3} \text{-} \frac{5}{27}\right)$. Increasing over (-x,0) 0 and $\left(\frac{4}{3}, x\right)$. Decreasing over $\left(0, \frac{4}{3}\right)$. Point of inflection $\left(\frac{2}{3} - \frac{11}{27}\right)$. Concave down on $\left(-x, \frac{2}{3}\right)$ Concave up on $\left(\frac{2}{3}, x\right)$ Local maximum of 1 at x = 0. Local minimum of $-\frac{5}{27}$ at x = $\frac{4}{3}$.

2. You may use the graph given in question 1 to help you answer this question. The graph defined by $y = x^3\text{-}2^{x2} + 1$ is reflected about the x-axis. Give the defining equation of the reflected graph and its key features.

Answers: $y = x^3\text{-}2^{x2} \text{-}1$

For the graph see Figure 9.

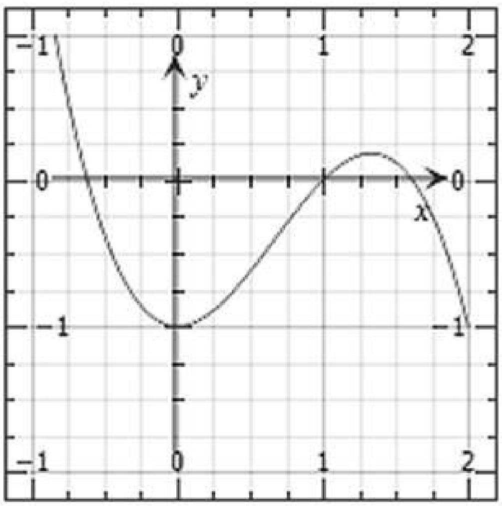

Fig. 9.

Domain is $(- x, x)$ and range is .(-x, x)y-intercept is -1.

x-intercepts are: x= 1 or $x = \frac{\sqrt{1}\text{-}5}{2}$ or $x = \frac{\sqrt{5}+1}{2}$

Turning points (0, 1) and $\left(-\frac{4}{3} - \frac{5}{27}\right)$. Decreasing over $\left(-x, \frac{2}{3}\right)$. Concave down on $\left(\frac{2}{3}, x\right)$

Local minimum of -1 at x=0. Local maximum of $-\frac{5}{27}$ at x = $\frac{4}{3}$

Local minimum of -1 at .Local maximum of at

.3.You may use the graph given in question 1 to help you answer this question. The graph defined by is reflected about the y-axis. Give the defining equation of the reflected graph and its key features.

Answers: $y = -x^3 - 2x^2 + 1$

For the graph see Figure 10.

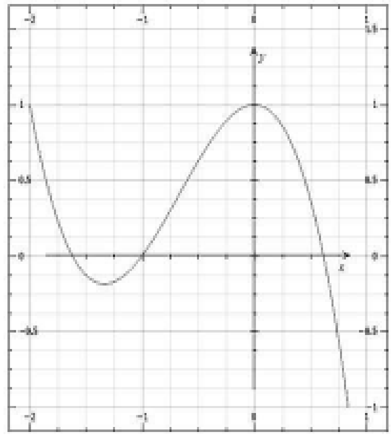

Fig. 10.

Domain is (-x, x) and range is (-x,x).y-intercept is 1.x-intercepts are: x = -1 or

$x = \frac{-1+\sqrt{5}}{2}$ or $x = \frac{\sqrt{5}\text{-}1}{2}$

Turning points (0, 1) and $\left(-\frac{4}{3} - \frac{5}{27}\right)$. Decreasing over $\left(-x, \frac{4}{3}\right)$U (0, x). Increasing over $\left(\frac{4}{3}, 0\right)$. Point of inflection $\left(-\frac{2}{3} - \frac{11}{27}\right)$. Concave up on $\left(\frac{2}{3}, x\right)$ Local minimum of $-\frac{5}{27}$ at x = $-\frac{4}{3}$

Exponential and Logarithmic Functions

Diagnostic questions (see Diagnostics 4) were formulated to make students observe and make conclusions on the key features of exponential and logarithmic functions, see questions 1 and 4. The questions were designed to increase student observations on the effect of translations or reflections (about an axis) on the key features of such functions, for examples see questions 2, 3, 5 and 6. This includes the effects of these (translations or reflections) on asymptotes and calculus related features. Students are also expected to be able to relate an exponential function to its inverse, see question 7.

Diagnostics 4: Diagnostic questions on exponential and logarithmic functions

1. Study the graph in Figure 11 of the function defined by . For this function identify the

intercepts, domain and range, asymptote(s), intervals of increase or decrease, and the interval(s) of concavity.

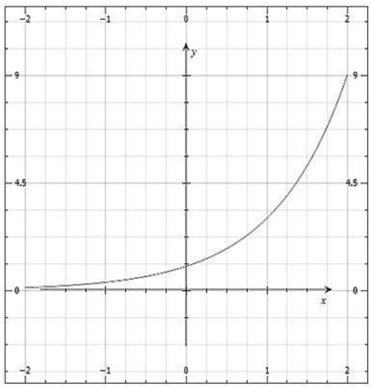

Fig. 11.

Answers: y-intercept is 1; domain is $\chi \varepsilon (\propto,\propto)$ and range is $y > 0$ horizontal asymptote x-axis defined by; increasing over $(-\propto,\propto)$; concave upward over $(-\propto,\propto)$.

2. Study the graph in Figure 12of the function defined by which is the result of a reflection and translation of a standard function. Identify the standard function and the reflection and translation on it that gives $y= x3^{-x}$. For the function identify the intercept(s), asymptotes, domain and range, intervals of increase or decrease, and the interval(s) of concavity.

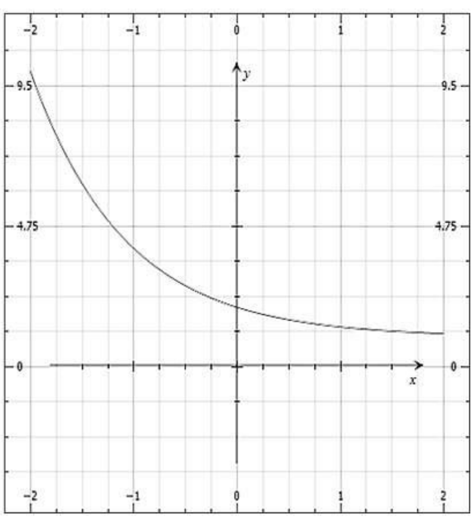

Fig. 12.

Answers: Standard function is $y=3^x$. Reflected about the y-axis and translated by 1 unit vertically upwards. For $y=1+3^{-x}$: y-intercept is 2; horizontal asymptote is defined by y=1; domain $(-\propto,\propto)$ and range $(1,\propto)$; decreasing on $(-\propto,\propto)$; concave upward on $(-\propto,\propto)$.

3. Study the defining equation of the function $y = 2\text{-}^{x-1}$ which is the result of a reflection and translation of the graph of a standard function. Identify the defining equation of the standard function, the required translation and reflection that gives the graph of the function . For the function identify the intercepts, domain and range, intervals of increase or decrease, and the intervals of concavity.

Answers: Standard function is $y=2^x$. Graph of $y=2^x$ translated horizontally to the left by 1 unit to get $y=2^{x+1}$, which is then reflected along the y-axis to get $y=2\text{-}(^{x+1})$.

For: 2^{-x-1},y-intercept is 1/2; domain $(-\propto,\propto)$ and range $(0,\propto)$; decreasing on $(-\propto,\propto)$; concave upwards on $(-\propto,\propto)$.

4. Study the graph in Figure 13 of the function defined by . For this function identify the intercept(s), the domain and range, interval(s) of increase or decrease, and interval(s) of concavity.

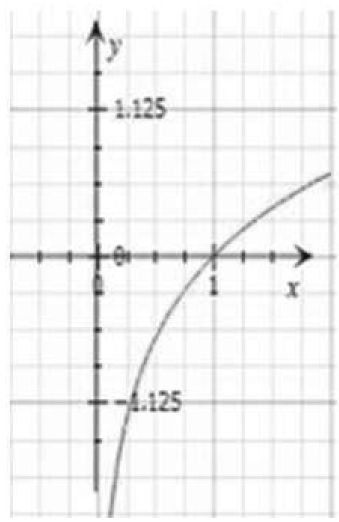

Fig. 13.

Answers: x-intercept is 1, domain is $(0,\propto)$ and range is$(-\propto,\propto)$, interval of increase $(0,\propto)$, concave downwards on $(0,\propto)$.

5. If the graph of $y=log_3x$ is reflected about the x-axis give the defining equation of the new graph and its key features.

Answers: $y=-log_3x$, x-intercept is 1, domain is $(0,\infty)$ and range is $(-\infty,\infty)$, vertical asymptote is the y-axis defined by $x=0$, decreasing on $(0,\infty)$, concave upwards on $(0,\infty)$.

6. If the graph of $y=-log_3x$, is reflected about the y-axis give the defining equation of the new graph and its key features.

Answers: $y=-log_3(-x)$, . x-intercept is -1, domain $(-0,\infty)$, range $(-\infty,\infty)$, vertical asymptote is the y-axis defined by , decreasing on $(-\infty,0)$, concave down on $(-\infty,0)$.

7. Give the defining equation of the inverse of the function defined by $y=3^x$ and identify its key features.

Answers: See question 4 and its answers.

Square Root and Cube Root Functions

Possibly such functions are new to some first year students, unless they studied an inverse of the function defined by $y=x^2$. Diagnostic questions were formulated (see Diagnostics5) to focus students' observations on the key features of square root and cube root functions, and the effect of translations and reflections along an axis, on their defining equations and resulting features. Notethat the focus on these questions is also on the structure of the standard square root and cube root functions, and respectively (see questions 1, 4 and 5), and effects of the above (translations and reflections) on the structure. The questions also focus on calculus related concepts, for example the concept of absolute extrema is also tested.

Diagnostics 5: Diagnostic questions and answers on square root and cube root functions

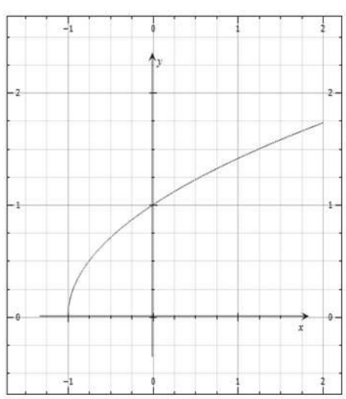

Fig. 14.

1. Study the graph in Figure 14 of the function defined by on a restricted domain. This graph is the result of a translation of a standard function. Identify the standard function and the translation that gives . For the function identify its key features.

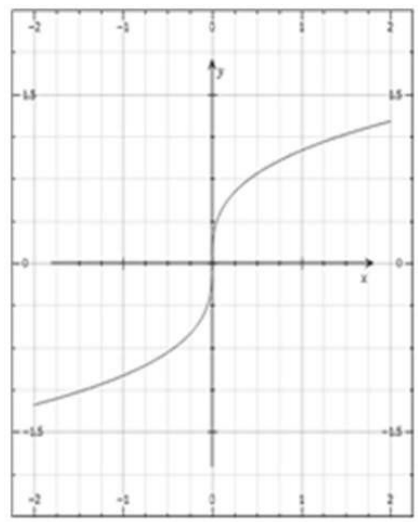

Fig. 15.

Answers: Standard function . The graph of is translated horizontally to the left by 1 unit. Domain and range . Increasing and concave down on its domain. Absolute minimum of 0 at .

2. If graph of the function is reflected along the y-axis give the defining equation and key features of the reflected graph.

Answers: Domain and range . Decreasing and concave down on its domain. Absolute minimum of 0 at .

3. If graph of the function is reflected along the x-axis give the defining equation and key features of the reflected graph.

Answers:

Domain and range. Decreasing and concave up on its domain. Absolute maximum of 0 at .

4. Study the graph in Figure 15 of the function defined by and identify its key features.

Answers: Domain and range. Increasing over its domain. Point of inflection . Concave up on and concave down on .

5. The graph of is translated by 1 unit horizontally to the left, moved vertically upwards by 2 units and then reflected about the x-axis. For

the graph that is finally arrived at give the defining equation and key features.

Answers:

For the graph see Figure 16.

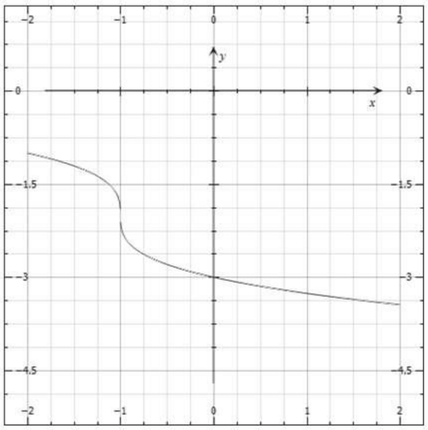

Fig. 16.

Domain and range .x-intercept is -9, y-intercept is -3.Decreasing over its domain.Point of inflection is .Concave down on , concave up on .

Building New Functions

Based on the identified learning outcomes the researchers formulated diagnostic questions to test whether students could use standard functions to build new functions. Those questions are indicated in Diagnostics 7 and 8. A study of the questions in Diagnostics 6 reveals that the designing of these questions focused on checking student knowledge and abilities in the context of translations of trigonometric functions (question 1), the inverse of a trigonometric function (question 2), use of trigonometric identities to simplify new functions arrived at (question 3), using the key features of a known function to determine the resulting features of its reciprocal function (question 4); on a restricted domain.

Diagnostics 6: Diagnostic questions and answers, building new trigonometric functions

Study the graph in Figure 17of the function defined by on a restricted domain. This graph is the result of a translation of a standard function. Identify the standard function and the translation that gives . For the function on the restricted domain identify its key features.

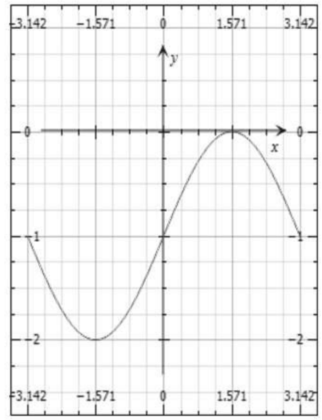

Fig. 17.

Answers: Standard function is . The graph of is translated vertically downwards by 1 unit. Restricted domain . Range . Decreasing on . Increasing on . Concave up on . Concave down on . Point of inflection . Absolute minimum of -2 at . Absolute maximum of 0 at .

1. Study the graph in Figure 18 of the function defined by and identify its key features.

Answers: Domain . Range . Increasing over its domain. Point of inflection (0,0). Concave down on and concave up on . Absolute maximum of at .

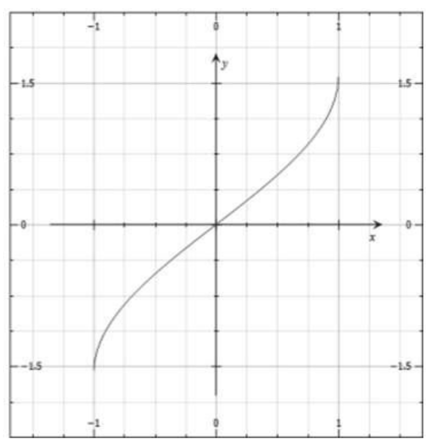

Fig. 18.

2. Multiply the functions and . Use your knowledge of trigonometric identities to express the defining equation in simpler form and state the key features of the graph of

Answers:

For the graph see Figure 19.

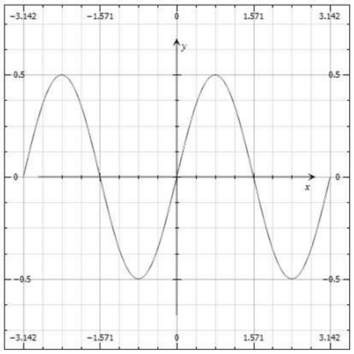

Fig. 19.

Domain and range .Periodic with period of , see sketch.On the restricted domain of the following occur:Increasing on .Decreasing on Pont of inflection .Concave up on .Concave down on.

Absolute maximum of at Absolute minimum of at

3.Use the graph of to determine the key features of the graph of for the restricted domain .

Answers:

Range . Vertical asymptotes are defined by .

Increasing on .Decreasing on .Concave up on .

Concave down on .Local minimum of 1 at . Local maximum of -1 at .

For the graph see Figure 20.

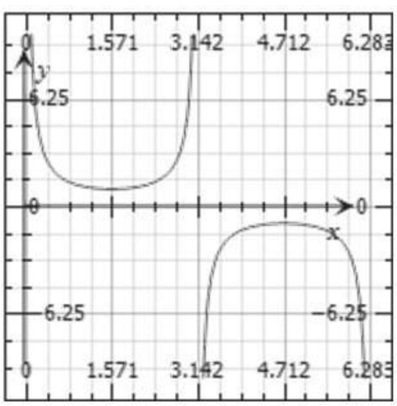

Fig. 20.

Diagnostic questions to check student technical knowledge and skills with regard to rational functions and the building of new functions are indicated in Diagnostics 7. The researchers' formulations of these questions focused on testing students on some features of a given graphical representation of a rational function (question 1). Observe that the defining equation enables students to make calculations for information that is not clear from the graph, for example the y-intercept and to verify vertical asymptotes. So it aids verification. Question 2 should evoke an appreciation of the importance of algebraic skills. In the literature review (Clark et al. 1997;Hassani 1998; Cottrill 1999) the importance of composition of functions was noted, so questions 3 and 4 test this aspect.

Diagnostics 7: Diagnostic questions for rational functions, building new functions

1. Study the graphical illustration in Figure 21 for a portion of the graph defined by the equation . For this function determine the y-intercept, the defining equations of the asymptotes, the intervals over which the graph is increasing or decreasing, the intervals of concavity, and the local extrema.

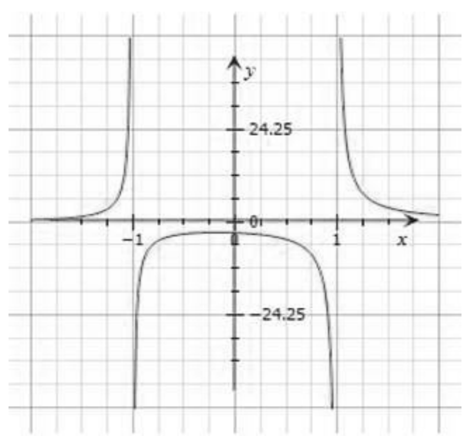

Fig. 21.

Answers: y-intercept is -3; vertical asymptotes are defined by , horizontal asymptote x-axis defined by ; increasing over , decreasing over ; concave up over ; local maximum of -3 at

2. Add the functions Give the key features of the graph of the new function.

Answers: New function . The key features are the same as that of the parabola , see the graph in Figure 22.

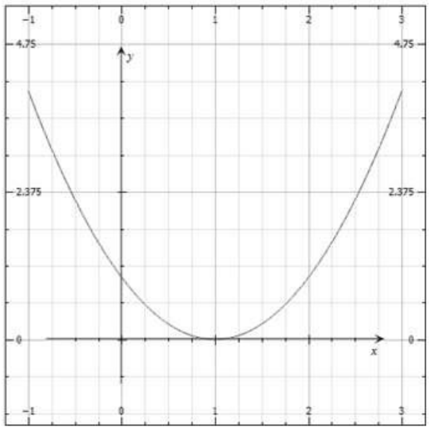

Fig. 22.

3. Give the composite function of and if and . Identify the key features of the graph of this composite function.

Answers:

for the domain . Observe that for this domain.

Range . Absolute minimum of 0 at . Increasing linearly on its domain.

4. Give the composite function of and if and . Identify the key features of the graph of this composite function.

Answers:

Domain . Range . Period , see sketch in Figure 23.

For the restricted domain the following can be observed:

Increasing on Decreasing on Points of inflection and .

Concave up on. Concave down on Absolute maximum of 1 at

Absolute minimum of 0 at

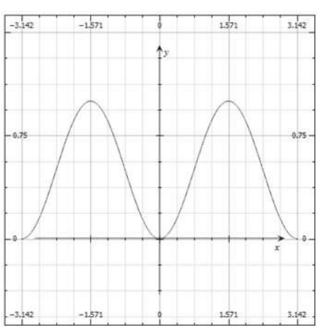

Fig. 23.

5. Consider the graph of the function . Identity the key features of the graph of the function defined by .

Answers:

For key features see the graph in Figure 24. Observe that the part of the graph of function below the x-axis is reflected along the x-axis to get the required graph.

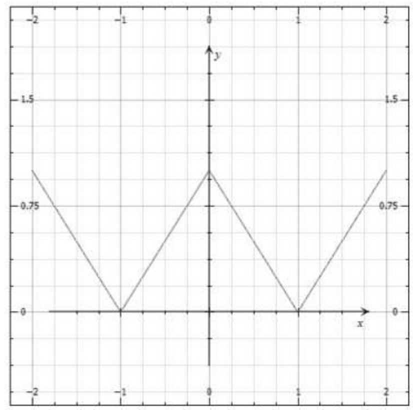

Fig. 24.

6. For the function identify the domain and comment on the behaviour of the function.

Answers:

Domain .Observe the graph in Figure 25. Note that for , the base of the function is raised to a negative number. For , the base of the function is based to a positive number.Vertical asymptote at , to the right of this the function decreases and then increases.Function is concave up on its domain.

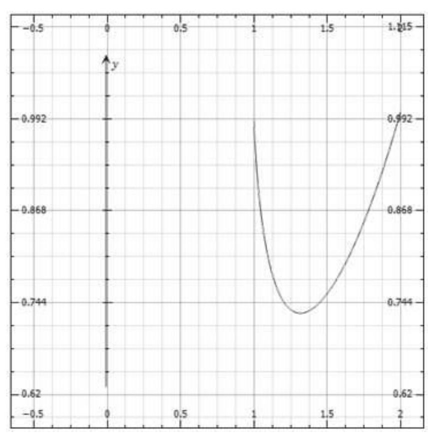

Fig. 25.

7. Study the function defined by and state its key features.

Answers: Domain . Range . Decreasing and concave up on , increasing and concave up on , and increasing and concave down on . Discontinuities at

You should sketch the function to see its features.

In Diagnostics 7 the diagnostic questions 5 to 7 were designed to check whether students could apply known knowledge and skills to new contexts, including the observation of discontinuities (question 7). The researchers believe that a student who attempts the diagnostics questions indicated in Diagnostics 2 to 7, and works on his/her weaknesses, could attain the learning outcomes formulated for functions. The questions were designed with suitable connections to focus students' learning about functions and their graphical interpretation (Mahir 2010), aimed at ensuring that students effectively learn concepts in calculus. Adding this visual dynamic to the analytical dimension of the teaching of the concepts should enable students to better understand functions and their calculus related concepts better. In their discussion the researchers repeatedly pointed out the focus of their approach is on the interrelationship between the visual and the analytical. The literature review indicates that Haciomeroglu et al. (2010) argued for a balanced reliance on verbal-logical (analytical) and visual-pictorial (visual) processes, and the formulations of the diagnostic questions took that into account. The procedure the researchers adopted in formulating the expected learning outcomesinformed the design of the sample diagnostic test questions. Further,both of these were informed by the literature review, so they should lead to improved student performance in differential calculus.

CONCLUSION

To develop the in-course diagnostics for functions in the context of the differential calculus module offered at UKZN, informed by the literature review the researchers formulated detailed learning outcomes for functions in the given context. This enabled the researchers to formulate diagnostic questions for different function types, as indicated in the findings and discussion section. These would have to be implemented either in the form of hardcopy written tests or web-based online diagnostic tests, to enable the researchers to check that their studentsachieve the expected learning outcomes timeously. Assumptions that the researchers made before embarking on the formulation of their diagnostics for functions were: (1) exposure to such questions would improve the observational capacity of students; (2) the implementation of the developed material would inform students of their strengths and weaknesses; (3) they would take appropriate measures to address their identified areas of weaknesses; and (4) the diagnostic materials would improve student performance. Next they will plan carefully for the implementation of their diagnostic tests on functions. This would enable them to investigate the attainment level of their assumptions, and also reflect on their learning outcomes and sample diagnostic tests for functions.

RECOMMENDATIONS

It is recommended that lecturers of first year mathematics modules should formulate learning outcomes for the section on functions. The learning outcomes formulated in this paper could be amended as they deem appropriate. Lecturers could also use or amend the sample diagnostic questions provided in this paper. Diagnostic tests should be given to their students in either a paper-based format or web-based format depending on the availability of resources. Once such tests are implemented their results should be analysed and reflected on. Those should provide substantial data and inform further research.

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